On d-graceful labelings

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Abstract

In this paper we introduce a generalization of the well known concept of a graceful labeling. Given a graph Γ with $e = d \cdot m$ edges, we call d-graceful labeling of Γ an injective function from $V(\Gamma)$ to the set $\{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that $\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m+1)-1\} - \{m+1, 2(m+1), \ldots, (d-1)(m+1)\}$. In the case of $d = 1$ and of $d = e$ we find the classical notion of a graceful labeling and of an odd graceful labeling, respectively. Also, we call d-graceful α-labeling of a bipartite graph Γ a d-graceful labeling of Γ with the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one. We show that these new concepts allow to obtain certain cyclic graph decompositions. We investigate the existence of d-graceful α-labelings for several classes of bipartite graphs, completely solving the problem for paths and stars and giving partial results about cycles of even length and ladders.

Keywords: graceful labeling; α-labeling; difference family; cyclic graph decomposition.
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1 Introduction

We will denote by K_v and $K_{m \times n}$ the *complete graph on v vertices* and the *complete m-partite graph with parts of size n*, respectively. For any graph Γ we write $V(\Gamma)$ for the set of its vertices, $E(\Gamma)$ for the set of its edges, and $D(\Gamma)$ for the set of its *di-edges*, namely the set of all ordered pairs (x, y) with x and y adjacent vertices of Γ. If $|E(\Gamma)| = e$, we say that Γ has *size e*.

Let Γ be a subgraph of a graph K. A Γ-*decomposition* of K is a set of graphs, called *blocks*, isomorphic to Γ whose edges partition $E(K)$. In the case that $K = K_v$ one also speaks of a Γ-*system of order v*. An *automorphism* of a Γ-decomposition $\mathcal{D}$ of K is a bijection on $V(K)$ leaving

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$\mathcal{D}$ invariant. A Γ-decomposition of K is said to be *cyclic* if it admits an automorphism consisting of a single cycle of length $|V(K)|$. In this case for giving the set $\mathcal{B}$ of blocks it is enough to give a complete system of representatives for the orbits of $\mathcal{B}$ under the cyclic group. The blocks of such a system are usually called *base blocks*.

For a survey on graph decompositions we refer to [3].

The concept of a *graceful labeling* of Γ introduced by A. Rosa [14] is quite related to the existence problem for cyclic Γ-systems. A *graceful labeling* of a graph Γ of size e is an injective map f from $V(\Gamma)$ to the set of integers $\{0, 1, 2, \ldots, e\}$ such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, \ldots, e\}.$$

The graph Γ is said to be *graceful* if there exists a graceful labeling of it. In the case that Γ is bipartite and f has the additional property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one, one says that f is an α-*labeling*.

Every graceful labeling of a graph Γ of size e gives rise to a cyclic Γ-system of order $2e + 1$ but an α-labeling of Γ gives much more; it gives in fact a cyclic Γ-system of order $2et + 1$ for every positive integer t (see [14]) and other kinds of graph decompositions such as, for instance, a *bicyclic* Γ-decomposition of $K_{2 \times e}$ (see [8]) and a cyclic Γ-decomposition of $K_{m \times e}$ for every odd integer m coprime with e (see [5]).

For a very rich survey on graceful labelings we refer to [9].

Many variations of graceful labelings are known. In particular Gnana Jothi [10] defines an *odd graceful labeling* of a graph Γ of size e as an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, 2e - 1\}$ such that

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 3, 5, \ldots, 2e - 1\}.$$

If such a function exists Γ is said to be *odd graceful*. She proved that every graph with an α-labeling is also odd graceful, while the converse, in general, is not true. One of the applications of these labelings is that trees of size e, with a suitable odd graceful labeling, can be used to generate cyclic decompositions of the complete bipartite graph $K_{2 \times e}$. For results on odd graceful graphs see [2, 7, 15].

In this paper we propose the following new definition which is, at the same time, a generalization of the concepts of a graceful labeling and of an odd graceful labeling.

Definition 1.1 *Let Γ be a graph of size e and let d be a divisor of e, say $e = d \cdot m$. A d-graceful labeling of Γ is an injective function $f : V(\Gamma) \to \{0, 1, 2, \ldots, d(m + 1) - 1\}$ such that*

$$\{|f(x) - f(y)| \mid [x, y] \in E(\Gamma)\} = \{1, 2, 3, \ldots, d(m + 1) - 1\}$$
$$- \{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}.$$

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If Γ admits a d-graceful labeling we will say that Γ is a *d-graceful graph*. Note that we find again the concepts of a graceful labeling and of an odd graceful labeling in the two extremal cases of $d = 1$ and $d = e$, respectively. By saying that d is *admissible* we will mean that it is a divisor of e and hence that it makes sense to investigate the existence of a d-graceful labeling of Γ.

It is well known that the complete graph K_v is graceful if and only if $v \leq 4$, see [11]. At the moment we are not able to extend this result to d-graceful labelings of the complete graph with $d > 1$. We recall that a (v, d, k, λ)-*relative difference set* (RDS) in a group G is a k-subset S of G such that the list of all differences $x - y$ with (x, y) an ordered pair of distinct elements of S covers $G - H$ exactly λ times, and no element of H at all with H a suitable subgroup of G of order d (see [13] for a survey on this topic). A RDS as above is said to be cyclic if G is such. It is immediate to recognize that every d-graceful labeling of K_v can be seen as a cyclic $(v(v-1) + 2d, 2d, v, 1)$-RDS though the converse is not true in general. For instance, the labeling of the vertices of K_5 with the elements of the set $S = \{0, 1, 4, 9, 11\}$ is 2-graceful and we see that S is a $(20, 4, 5, 1)$-RDS. Now note that $S' = \{0, 1, 3, 11, 20\}$ is also a cyclic $(20, 4, 5, 1)$-RDS, but it is evident that the labeling of the vertices of K_5 with the elements of S' is not 2-graceful. The very few known results about cyclic RDSs do not allow us to say much more about the d-gracefulness of the complete graph with $d > 1$.

The α-labelings can be generalized in a similar way.

Definition 1.2 *A d-graceful α-labeling of a bipartite graph Γ is a d-graceful labeling of Γ having the property that its maximum value on one of the two bipartite sets does not reach its minimum value on the other one.*

In the next section we will see how these new concepts can be used to obtain some graph decompositions. Then we investigate the existence of d-graceful α-labelings for several classes of bipartite graphs. In particular, we prove that paths and stars admit a d-graceful α-labeling for any admissible d. We also present partial results about d-graceful labelings of even cycles and ladders.

2 d-graceful labelings and graph decompositions

It is known that graceful labelings are related to difference families, see [1]. The concept of a *relative difference set* has been generalized in [4] to that

of a *relative difference family*. A further generalization has been introduced in [6].

Definition 2.1 *Let Γ be a graph and let d be a divisor of v. A $(v, d, \Gamma, 1)$-difference family (DF for short) is a collection $\mathcal{F}$ of injective maps from $V(\Gamma)$ to $\mathbb{Z}_v$ such that the list*

$$\Delta\mathcal{F} = \{f(x) - f(y) \mid f \in \mathcal{F}; \ (x, y) \in D(\Gamma)\}$$

covers $\mathbb{Z}_v - \frac{v}{d}\mathbb{Z}_v$ exactly once while it does not contain any element of $\frac{v}{d}\mathbb{Z}_v$, where $\frac{v}{d}\mathbb{Z}_v$ denotes the subgroup of $\mathbb{Z}_v$ of order d.

If a $(v, d, \Gamma, 1)$-DF consists of a single map f one says that f is a *difference graph*. In [6] it is proved that

Theorem 2.2 *If there exists a $(v, d, \Gamma, 1)$-DF then there exists a cyclic Γ-decomposition of $K_{\frac{v}{d} \times d}$.*

The next easy proposition establishes the link between d-graceful labelings and DFs.

Proposition 2.3 *A d-graceful labeling of a graph Γ of size e determines a $\left(2d(\frac{e}{d} + 1), 2d, \Gamma, 1\right)$-DF.*

Proof. Let $f : V(\Gamma) \to \left\{0, 1, \ldots, d\left(\frac{e}{d} + 1\right) - 1\right\}$ be a d-graceful labeling of Γ. It is obvious that $\phi : x \in V(\Gamma) \to f(x) \in \mathbb{Z}_{2d\left(\frac{e}{d} + 1\right)}$ is a $\left(2d(\frac{e}{d} + 1), 2d, \Gamma, 1\right)$-DF. $\qquad\qquad\square$

By the above considerations we can state

Proposition 2.4 *If there exists a d-graceful labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{\left(\frac{e}{d} + 1\right) \times 2d}$.*

Example 2.5 *Figure 1 shows a 2-graceful labeling of C_6. Hence $B = (0, 2, 3, 6, 1, 7)$ is a base block of a cyclic C_6-decomposition of $K_{4 \times 4}$.*

The following result shows that, as in the case of classical graceful labelings, d-graceful α-labelings are more powerful than d-graceful labelings.

Theorem 2.6 *If there exists a d-graceful α-labeling of a graph Γ of size e then there exists a cyclic Γ-decomposition of $K_{\left(\frac{e}{d} + 1\right) \times 2dn}$ for any integer $n \geq 1$.*

Proof. Let f be a d-graceful α-labeling of Γ, so that we have $\max f(X) < \min f(Y)$ where X and Y are the two parts of Γ. It is easy to see that the set of maps $\{f_1, f_2, \ldots, f_n\}$ from $V(\Gamma)$ to $\mathbb{Z}_{2dn\left(\frac{e}{d} + 1\right)}$ defined by

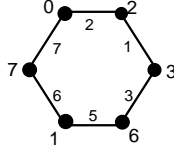$$f_i(x) = f(x) \qquad \forall x \in X, \ \forall i = 1, \ldots, n$$

Figure 1: A 2-graceful labeling of C_6

$$f_i(y) = f(y) + (i-1)(d+e) \qquad \forall y \in Y, \ \forall i = 1, \ldots, n$$

is a $\left(2dn(\frac{e}{d}+1), 2dn, \Gamma, 1\right)$-DF. Hence by Theorem 2.2 there exists a cyclic Γ-decomposition of $K_{\left(\frac{e}{d}+1\right) \times 2dn}$ for any integer $n \geq 1$. $\qquad \square$

Example 2.7 *Figure 2 shows a 2-graceful α-labeling of C_6 such that there exists a cyclic C_6-decomposition of $K_{4 \times 4n}$ for any integer n, $n \geq 1$. The set of base blocks of such a decomposition is given by*

$$\{(0, 5+8i, 2, 3+8i, 1, 7+8i) \mid i = 0, \ldots, n-1\}.$$

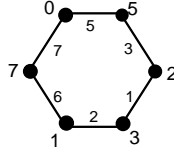Figure 2: A 2-graceful α-labeling of C_6

3 d-graceful α-labelings of paths and stars

In this section we will show that paths and stars have a d-graceful α-labeling for any admissible d. From now on, given two integers a and b, by $[a, b]$ we will denote the set of integers x such that $a \leq x \leq b$.

As usual we will denote by P_{e+1} the path on $e+1$ vertices, having size e. We recall that any path admits an α-labeling, see [14].

Theorem 3.1 *Given a positive integer e, the path P_{e+1} has a d-graceful α-labeling for any admissible d.*

Proof. Let $e = d \cdot m$ and let $P_{dm+1} = (x_1 \sim x_2 \sim \ldots \sim x_{dm+1})$. We distinguish two cases depending on the parity of m.

Case 1: m even.
Denote by $\mathcal{O} = \{x_1, x_3, \ldots, x_{dm+1}\}$ and $\mathcal{E} = \{x_2, x_4, \ldots, x_{dm}\}$ the bipartite sets of P_{dm+1}. Let $f : V(P_{dm+1}) \to [0, d(m+1) - 1]$ be defined as follows:

$$f(x_{2i+1}) = i \quad \text{for } i \in \left[0, \frac{dm}{2}\right]$$

$$f(x_{2i}) = \begin{cases} d(m+1) - i & \text{for } i \in \left[1, \frac{m}{2}\right] \\ d(m+1) - i - 1 & \text{for } i \in \left[\frac{m}{2} + 1, m\right] \\ d(m+1) - i - 2 & \text{for } i \in \left[m+1, \frac{3m}{2}\right] \\ \vdots & \vdots \\ d(m+1) - i - (d-1) & \text{for } i \in \left[\frac{(d-1)m}{2} + 1, \frac{dm}{2}\right]. \end{cases}$$

One can check that

$$f(\mathcal{O}) = \left[0, \frac{dm}{2}\right]$$

$$f(\mathcal{E}) = \left[\frac{dm}{2} + 1, \frac{(d+1)m}{2}\right] \cup \left[\frac{(d+1)m}{2} + 2, \frac{(d+2)m}{2} + 1\right] \cup$$

$$\cup \left[\frac{(d+2)m}{2} + 3, \frac{(d+3)m}{2} + 2\right] \cup \ldots \cup \left[d(m+1) - \frac{m}{2}, d(m+1) - 1\right].$$

Hence f is injective and $\max f(\mathcal{O}) < \min f(\mathcal{E})$. Now set

$$\varepsilon_i = |f(x_{2i-1}) - f(x_{2i})|, \quad \rho_i = |f(x_{2i}) - f(x_{2i+1})| \quad \text{for } i = 1, \ldots, \frac{dm}{2}. \tag{1}$$

By a direct calculation, one can see that

$$\left\{\varepsilon_i, \rho_i \mid i = 1, \ldots, \frac{m}{2}\right\} = [(d-1)(m+1) + 1, d(m+1) - 1];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{m}{2} + 1, \ldots, m\right\} = [(d-2)(m+1) + 1, (d-1)(m+1) - 1];$$

$$\ldots$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{(d-2)m}{2} + 1, \ldots, \frac{(d-1)m}{2}\right\} = [m+2, 2(m+1) - 1];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{(d-1)m}{2} + 1, \ldots, \frac{dm}{2}\right\} = [1, m].$$

This concludes case 1.

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Case 2: m odd.

We divide this case in two subcases depending on the parity of d.

Case 2a): d odd. In this case the bipartite sets are $\mathcal{O} = \{x_1, x_3, \ldots, x_{dm}\}$ and $\mathcal{E} = \{x_2, x_4, \ldots, x_{dm+1}\}$. Let $f : V(P_{dm+1}) \to [0, d(m+1)-1]$ be defined as follows:

$$
f(x_{2i+1}) = \begin{cases}
i & \text{for } i \in \left[0, \frac{m-1}{2}\right] \\
i+1 & \text{for } i \in \left[\frac{m+1}{2}, \frac{3m-1}{2}\right] \\
i+2 & \text{for } i \in \left[\frac{3m+1}{2}, \frac{5m-1}{2}\right] \\
\vdots & \vdots \\
i+\frac{d-1}{2} & \text{for } i \in \left[\frac{(d-2)m+1}{2}, \frac{dm-1}{2}\right].
\end{cases}
$$

$$
f(x_{2i}) = \begin{cases}
d(m+1)-i & \text{for } i \in [1, m] \\
d(m+1)-i-1 & \text{for } i \in [m+1, 2m] \\
d(m+1)-i-2 & \text{for } i \in [2m+1, 3m] \\
\vdots & \vdots \\
d(m+1)-i-\frac{d-1}{2} & \text{for } i \in \left[\frac{(d-1)m}{2}+1, \frac{dm+1}{2}\right].
\end{cases}
$$

In this case it results that

$$
f(\mathcal{O}) = \left[0, \frac{m-1}{2}\right] \cup \left[\frac{m+3}{2}, \frac{3m+1}{2}\right] \cup
$$
$$
\cup \left[\frac{3m+5}{2}, \frac{5m+3}{2}\right] \cup \ldots \cup \left[\frac{(d-2)m+d}{2}, \frac{d(m+1)}{2}-1\right]
$$
$$
f(\mathcal{E}) = \left[\frac{d(m+1)}{2}, \frac{(d+1)(m+1)}{2}-1\right] \cup
$$
$$
\cup \left[\frac{(d+1)(m+1)}{2}+1, \frac{(d+3)(m+1)}{2}-1\right] \cup
$$
$$
\cup \left[\frac{(d+3)(m+1)}{2}+1, \frac{(d+5)(m+1)}{2}-1\right] \cup \ldots
$$
$$
\ldots \cup [d(m+1)-m, d(m+1)-1].
$$

So, also in this case, f is injective and $\max f(\mathcal{O}) < \min f(\mathcal{E})$. Note that, now, since dm is odd we have to consider

$$
\varepsilon_i = |f(x_{2i-1}) - f(x_{2i})|, \qquad \rho_i = |f(x_{2i}) - f(x_{2i+1})| \quad \text{for } i = 1, \ldots, \frac{dm-1}{2},
$$
$$
\varepsilon_{\frac{dm+1}{2}} = |f(x_{dm}) - f(x_{dm+1})|.
$$

By a long and tedious calculation, one can see that

$$\left\{\varepsilon_i, \rho_i \mid i = 1, \ldots, \frac{m-1}{2}\right\} \cup \left\{\varepsilon_{\frac{m+1}{2}}\right\} = [(d-1)(m+1)+1,$$
$$d(m+1) - 1];$$

$$\left\{\rho_{\frac{m+1}{2}}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{m+3}{2}, \ldots, m\right\} = [(d-2)(m+1)+1,$$
$$(d-1)(m+1) - 1];$$

$$\ldots$$

$$\left\{\rho_{\frac{(d-2)m+1}{2}}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{(d-2)m+1}{2} + 1, \ldots, \frac{(d-1)m}{2}\right\} =$$
$$= [m+2, 2(m+1) - 1];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{(d-1)m}{2} + 1, \ldots, \frac{dm-1}{2}\right\} \cup \left\{\varepsilon_{\frac{dm+1}{2}}\right\} = [1, m].$$

This completes the proof of Case 2a).

Case 2b): d even. Let $f : V(P_{dm+1}) \rightarrow [0, d(m+1) - 1]$ be defined as follows:

$$f(x_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{m-1}{2}\right] \\ i+1 & \text{for } i \in \left[\frac{m+1}{2}, \frac{3m-1}{2}\right] \\ i+2 & \text{for } i \in \left[\frac{3m+1}{2}, \frac{5m-1}{2}\right] \\ \vdots & \vdots \\ i + \frac{d}{2} & \text{for } i \in \left[\frac{(d-1)m+1}{2}, \frac{dm}{2}\right] \end{cases}$$

$$f(x_{2i}) = \begin{cases} d(m+1) - i & \text{for } i \in [1, m] \\ d(m+1) - i - 1 & \text{for } i \in [m+1, 2m] \\ d(m+1) - i - 2 & \text{for } i \in [2m+1, 3m] \\ \vdots & \vdots \\ d(m+1) - i - \frac{d-2}{2} & \text{for } i \in \left[\frac{(d-2)m}{2} + 1, \frac{dm}{2}\right]. \end{cases}$$

Also here, reasoning as in Case 2a), it is easy to see that f is a d-graceful α-labeling of P_{dm+1}. $\qquad\square$

Proposition 3.2 *There exists a cyclic P_{e+1}-decomposition of $K_{\left(\frac{e}{d}+1\right) \times 2dn}$ for any integers $e, n \geq 1$ and any divisor d of e.*

Proof. It is a direct consequence of theorems 2.6 and 3.1. □

In Figure 3 we have the d-graceful α-labelings of P_{19} for $d = 2, 3, 6, 9, 18$, obtained following the construction given in the proof of Theorem 3.1. Note that these are all the possible d-graceful α-labelings of P_{19}, apart from the already known case of a classical α-labeling ($d = 1$), [14].
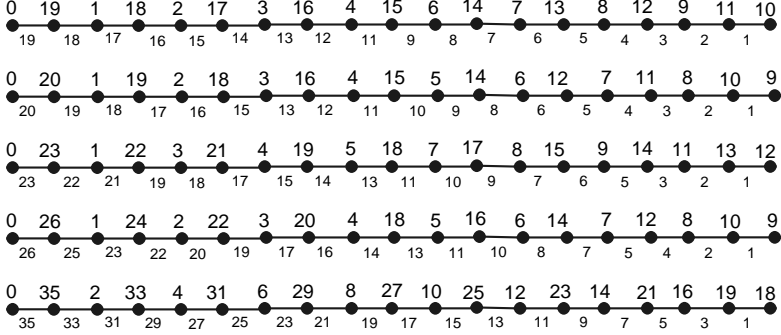

Figure 3: P_{19}

We recall that the *star* on $e + 1$ vertices S_{e+1} is the complete bipartite graph with one part having a single vertex, called the *center* of the star, and the other part having e vertices, called *external vertices*. It is obvious that any star has an α-labeling.

Theorem 3.3 *Given a positive integer e, the star S_{e+1} has a d-graceful α-labeling for any admissible d.*

Proof. Let $e = d \cdot m$. It is obvious that if we label the center of the star with 0 and the external vertices, arbitrarily, with the elements of $\{1, 2, 3, \ldots, d(m + 1) - 1\} - \{m + 1, 2(m + 1), \ldots, (d - 1)(m + 1)\}$, we have a d-graceful α-labeling of S_{e+1}. □

Applying Theorem 2.6 we have:

Proposition 3.4 *There exists a cyclic S_{e+1}-decomposition of $K_{\left(\frac{e}{d}+1\right) \times 2dn}$ for any integers $e, n \geq 1$ and any divisor d of e.*

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4 d-graceful α-labelings of cycles and ladders

As usual, we will denote by C_k the cycle on k vertices. It is obvious that C_k is a graph of size k and that it is bipartite if and only if k is even. In [14] Rosa proved that C_k has an α-labeling if and only if $k \equiv 0 \pmod 4$.

Theorem 4.1 *The cycle C_{4k} has a 2-graceful α-labeling for every $k \geq 1$.*

Proof. Let $C_{4k} = (x_1, x_2, \ldots, x_{4k})$ and denote by $\mathcal{O} = \{x_1, x_3, \ldots, x_{4k-1}\}$ and $\mathcal{E} = \{x_2, x_4, \ldots, x_{4k}\}$ its bipartite sets of vertices. Consider the map: $f : V(C_{4k}) \to \{0, 1, \ldots, 4k+1\}$ defined as follows:

$$f(x_{2i+1}) = i \quad \text{for } i \in [0, 2k-1]$$

$$f(x_{2i}) = \begin{cases} 4k + 2 - i & \text{for } i \in [1, k] \\ 4k - i & \text{for } i \in [k+1, 2k]. \end{cases}$$

We have

$$f(\mathcal{O}) = [0, 2k-1] \quad f(\mathcal{E}) = [2k, 3k-1] \cup [3k+2, 4k+1].$$

In this way we see that f is injective. Also, note that $\max f(\mathcal{O}) < \min f(\mathcal{E})$. Now set ε_i, ρ_i (for $i = 1, \ldots, 2k$) as in (1), where the indices are understood modulo $4k$. One can easily check that

$$\{\varepsilon_i, \rho_i \mid i = 1, \ldots, k\} = [2k+2, 4k+1]$$

and

$$\{\varepsilon_i, \rho_i \mid i = k+1, \ldots, 2k\} = [1, 2k].$$

Hence f is a 2-graceful α-labeling of C_{4k}. $\qquad\qquad \square$

Theorem 4.2 *The cycle C_{4k} admits a 4-graceful α-labeling for every $k \geq 1$.*

Proof. Let $\mathcal{O}$ and $\mathcal{E}$ be defined as in the proof of the previous theorem. We are able to prove the existence of a 4-graceful α-labeling of C_{4k} by means of two direct constructions where we distinguish the two cases: k even and k odd.

Case 1: k even.
Consider the map: $f : V(C_{4k}) \to \{0, 1, \ldots, 4k+3\}$ defined as follows:

$$f(x_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{3k}{2} - 1\right] \\ i + 1 & \text{for } i \in \left[\frac{3k}{2}, 2k - 1\right] \end{cases}$$

$$f(x_{2i}) = \begin{cases} 4k + 4 - i & \text{for } i \in \left[1, \frac{k}{2}\right] \\ 4k + 3 - i & \text{for } i \in \left[\frac{k}{2} + 1, k\right] \\ 4k + 1 - i & \text{for } i \in [k + 1, 2k]. \end{cases}$$

It is easy to see that

$$f(\mathcal{O}) = \left[0, \frac{3k}{2} - 1\right] \cup \left[\frac{3k}{2} + 1, 2k\right]$$

$$f(\mathcal{E}) = [2k + 1, 3k] \cup \left[3k + 3, \frac{7k}{2} + 2\right] \cup \left[\frac{7k}{2} + 4, 4k + 3\right].$$

Hence f is injective and $\max f(\mathcal{O}) < \min f(\mathcal{E})$. Set ε_i and ρ_i (for $i = 1, \ldots, 2k$) as in (1), where the indices are understood modulo $4k$. It is not hard to see that

$$\left\{\varepsilon_i, \rho_i \mid i = 1, \ldots, \frac{k}{2}\right\} = [3k + 4, 4k + 3];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{k}{2} + 1, \ldots, k\right\} = [2k + 3, 3k + 2];$$

$$\left\{\varepsilon_i, \rho_i \mid i = k + 1, \ldots, \frac{3k}{2} - 1\right\} = [k + 3, 2k];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{3k}{2}, \ldots, 2k\right\} = [1, k] \cup \{k + 2\} \cup \{2k + 1\}.$$

The thesis follows.

Case 2: k odd.
Let now $f : V(C_{4k}) \to \{0, 1, \ldots, 4k + 3\}$ be defined as follows:

$$f(x_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{k-1}{2}\right] \\ i + 1 & \text{for } i \in \left[\frac{k+1}{2}, 2k - 1\right] \end{cases}$$

$$f(x_{2i}) = \begin{cases} 4k + 4 - i & \text{for } i \in [1, k] \\ 4k + 2 - i & \text{for } i \in \left[k + 1, \frac{3k-1}{2}\right] \\ 4k + 1 - i & \text{for } i \in \left[\frac{3k+1}{2}, 2k\right]. \end{cases}$$

One can check that

$$f(\mathcal{O}) = \left[0, \frac{k-1}{2}\right] \cup \left[\frac{k+3}{2}, 2k\right]$$

$$f(\mathcal{E}) = \left[2k + 1, \frac{5k+1}{2}\right] \cup \left[\frac{5k+5}{2}, 3k + 1\right] \cup [3k + 4, 4k + 3].$$

Also in this case f is injective and $\max f(\mathcal{O}) < \min f(\mathcal{E})$. By a direct calculation, one can see that

$$\left\{\varepsilon_i, \rho_i \mid i = 1, \ldots, \frac{k-1}{2}\right\} = [3k+5, 4k+3];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{k+1}{2}, \ldots, k\right\} = [2k+3, 3k+2] \cup \{3k+4\};$$

$$\left\{\varepsilon_i, \rho_i \mid i = k+1, \ldots, \frac{3k-1}{2}\right\} = [k+2, 2k];$$

$$\left\{\varepsilon_i, \rho_i \mid i = \frac{3k+1}{2}, \ldots, 2k\right\} = [1, k] \cup \{2k+1\}.$$

The assertion then follows. $\qquad\square$

The two figures in Figure 4 show the 2-graceful α-labeling and the 4-graceful α-labeling of C_{16} provided by the previous theorems.
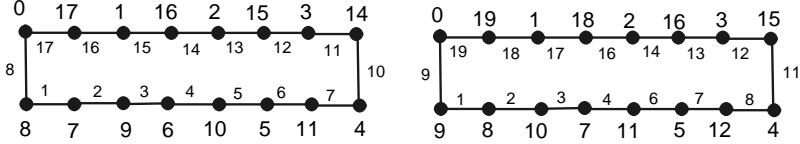

Figure 4: C_{16}

By virtue of Theorems 2.6, 4.1 and 4.2, we have

Proposition 4.3 *There exists a cyclic C_{4k}-decomposition of $K_{(2k+1)\times 4n}$, and a cyclic C_{4k}-decomposition of $K_{(k+1)\times 8n}$ for any integers $k, n \geq 1$.*

Theorem 4.4 *For any odd integer $k > 1$, the cycle C_{2k} is 2-graceful.*

Proof. For $k = 3, 5, 7$ we show directly a 2-graceful labeling:

$$C_6 = (0, 5, 2, 3, 1, 7),$$
$$C_{10} = (0, 11, 1, 3, 7, 4, 5, 10, 2, 9),$$
$$C_{14} = (0, 15, 1, 14, 11, 4, 10, 5, 7, 6, 2, 13, 3, 12).$$

Let now k be greater than 7 and set $C_{2k} = (x_1, x_2, x_3, \ldots, x_{2k})$. Let $\mathcal{O} = \{x_1, x_3, \ldots, x_{2k-1}\}$ and $\mathcal{E} = \{x_2, x_4, \ldots, x_{2k}\}$. We will construct a 2-graceful labeling of C_{2k} where we distinguish two cases according to whether $k \equiv 1 (\mathrm{mod}\ 4)$ or $k \equiv 3 (\mathrm{mod}\ 4)$.

Case 1: $k \equiv 1 \pmod 4$.

In this case $k = 2t + 1$ with $t \geq 4$ even. Consider the map $f : V(C_{2k}) \to \{0, 1, \ldots, 2k + 1\}$ defined as follows:

$$f(x_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{t}{2}\right] \\ \frac{7}{2}t + 3 - i & \text{for } i \in \left[\frac{t}{2} + 1, t\right] \\ 3t + 2 - i & \text{for } i \in \left[t + 1, t + \lfloor \frac{t}{4} \rfloor\right] \\ t + 2 + i & \text{for } i \in \left[t + \lfloor \frac{t}{4} \rfloor + 1, \frac{3}{2}t\right] \\ i - t & \text{for } i \in \left[\frac{3}{2}t + 1, 2t\right] \end{cases}$$

$$f(x_{2i}) = \begin{cases} 4t + 4 - i & \text{for } i \in \left[1, \frac{t}{2}\right] \\ \frac{1}{2}t + i & \text{for } i \in \left[\frac{t}{2} + 1, t + 1\right] \\ t + 1 + i & \text{for } i \in \left[t + 2, t + \lfloor \frac{t+2}{4} \rfloor\right] \\ 3t + 2 - i & \text{for } i \in \left[t + \lfloor \frac{t+2}{4} \rfloor + 1, \frac{3}{2}t\right] \\ 5t + 4 - i & \text{for } i \in \left[\frac{3}{2}t + 1, 2t + 1\right]. \end{cases}$$

If $t = 4$ skip the third assignment of $f(x_{2i})$.

One can directly check that

$$f(\mathcal{O}) = \left[0, \frac{t}{2}\right] \cup \left[\frac{5}{2}t + 3, 3t + 2\right] \cup \left[2t + 2 - \left\lfloor \frac{t}{4} \right\rfloor, 2t + 1\right] \cup$$

$$\cup \left[2t + 3 + \left\lfloor \frac{t}{4} \right\rfloor, \frac{5}{2}t + 2\right] \cup \left[\frac{t}{2} + 1, t\right]$$

$$f(\mathcal{E}) = \left[\frac{7}{2}t + 4, 4t + 3\right] \cup \left[t + 1, \frac{3}{2}t + 1\right] \cup \left[2t + 3, 2t + 1 + \left\lfloor \frac{t+2}{4} \right\rfloor\right] \cup$$

$$\cup \left[\frac{3}{2}t + 2, 2t + 1 - \left\lfloor \frac{t+2}{4} \right\rfloor\right] \cup \left[3t + 3, \frac{7}{2}t + 3\right].$$

So f is an injective function. Now set ε_i and ρ_i (for $i = 1, \ldots, k$) as in (1) where the indices are understood modulo $2k$.

By a direct calculation one can see that

$$\left\{\varepsilon_i, \rho_i \mid i = 1, \ldots, \frac{t}{2}\right\} = [3t + 4, 4t + 3]$$

$$\varepsilon_{\frac{t}{2}+1} = \frac{t}{2} + 1$$

$$\left\{\rho_{\frac{t}{2}+1}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{t}{2} + 2, \ldots, t\right\} \cup \{\varepsilon_{t+1}\} = [t + 2, 2t + 1]$$

$$\rho_{t+1} = \frac{t}{2}.$$

We write the remaining differences and distinguish two subcases depending on the parity of $\frac{t}{2}$. If $\frac{t}{2}$ is even, it results:

$$\left\{\varepsilon_i, \rho_i \mid i = t+2, \ldots, \frac{5}{4}t\right\} = \left[2, \frac{t}{2} - 1\right]$$

$$\varepsilon_{\frac{5}{4}t+1} = 1$$

$$\left\{\rho_{\frac{5}{4}t+1}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{5}{4}t + 2, \ldots, \frac{3}{2}t\right\} \cup \left\{\varepsilon_{\frac{3}{2}t+1}\right\} = \left[\frac{t}{2} + 2, t + 1\right]$$

$$\left\{\rho_{\frac{3}{2}t+1}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{3}{2}t + 2, \ldots, 2t\right\} \cup \left\{\varepsilon_{2t+1}\right\} = [2t + 3, 3t + 2]$$

$$\rho_{2t+1} = 3t + 3.$$

If $\frac{t}{2}$ is odd, we have:

$$\left\{\varepsilon_i, \rho_i \mid i = t+2, \ldots, \left\lfloor \frac{5}{4}t \right\rfloor\right\} \cup \left\{\varepsilon_{\lfloor \frac{5}{4}t \rfloor + 1}\right\} = \left[2, \frac{t}{2} - 1\right]$$

$$\rho_{\lfloor \frac{5}{4}t \rfloor + 1} = 1$$

$$\left\{\varepsilon_i, \rho_i \mid i = \left\lfloor \frac{5}{4}t \right\rfloor + 2, \ldots, \frac{3}{2}t\right\} \cup \left\{\varepsilon_{\frac{3}{2}t+1}\right\} = \left[\frac{t}{2} + 2, t + 1\right]$$

$$\left\{\rho_{\frac{3}{2}t+1}\right\} \cup \left\{\varepsilon_i, \rho_i \mid i = \frac{3}{2}t + 2, \ldots, 2t\right\} \cup \left\{\varepsilon_{2t+1}\right\} = [2t + 3, 3t + 2]$$

$$\rho_{2t+1} = 3t + 3.$$

This concludes the proof of case 1.

Case 2: $k \equiv 3 \pmod 4$.
In this case $k = 2t + 1$ with $t \geq 5$ odd. We define $f : V(C_{2k}) \to \{0, 1, \ldots, 2k + 1\}$ in the following way

$$f(x_{2i+1}) = \begin{cases} i & \text{for } i \in \left[0, \frac{t-1}{2}\right] \\ \frac{7t+5}{2} - i & \text{for } i \in \left[\frac{t+1}{2}, t\right] \\ 3t + 2 - i & \text{for } i \in \left[t+1, t + \lfloor \frac{t+1}{4} \rfloor\right] \\ t + 2 + i & \text{for } i \in \left[t + \lfloor \frac{t+1}{4} \rfloor + 1, \frac{3t-1}{2}\right] \\ i - t & \text{for } i \in \left[\frac{3t+1}{2}, 2t\right] \end{cases}$$

$$f(x_{2i}) = \begin{cases} 4t + 4 - i & \text{for } i \in \left[1, \frac{t+1}{2}\right] \\ \frac{t-1}{2} + i & \text{for } i \in \left[\frac{t+3}{2}, t + 1\right] \\ t + 1 + i & \text{for } i \in \left[t + 2, t + \lfloor \frac{t+3}{4} \rfloor\right] \\ 3t + 2 - i & \text{for } i \in \left[t + \lfloor \frac{t+3}{4} \rfloor + 1, \frac{3t+1}{2}\right] \\ 5t + 4 - i & \text{for } i \in \left[\frac{3t+3}{2}, 2t + 1\right]. \end{cases}$$

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Also in this case one can directly check that

$$f(\mathcal{O}) = \left[0, \frac{t-1}{2}\right] \cup \left[\frac{5t+5}{2}, 3t+2\right] \cup \left[2t+2 - \left\lfloor\frac{t+1}{4}\right\rfloor, 2t+1\right] \cup$$

$$\cup \left[2t+2 + \left\lfloor\frac{t+1}{4}\right\rfloor, \frac{5t+3}{2}\right] \cup \left[\frac{t+1}{2}, t\right]$$

$$f(\mathcal{E}) = \left[\frac{7t+7}{2}, 4t+3\right] \cup \left[t+1, \frac{3t+1}{2}\right] \cup \left[2t+3, 2t+1 + \left\lfloor\frac{t+3}{4}\right\rfloor\right] \cup$$

$$\cup \left[\frac{3t+3}{2}, 2t+2 - \left\lfloor\frac{t+3}{4}\right\rfloor\right] \cup \left[3t+3, \frac{7t+5}{2}\right].$$

By long calculations one can verify that, also in case 2, f is a 2-labeling of C_{2k}. $\qquad\square$

In Figure 5 we have the 2-graceful labeling of C_{26} and that of C_{30} obtained through the construction given in the proof of Theorem 4.4.
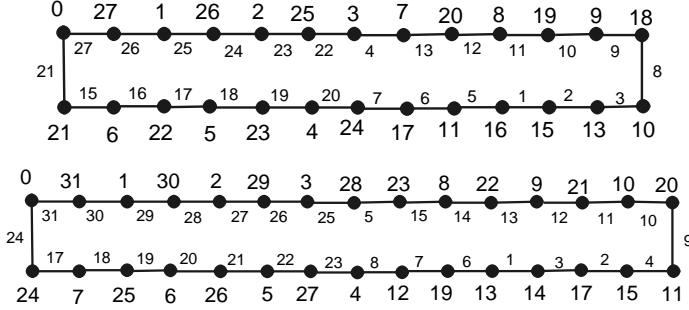

Figure 5: C_{26} and C_{30}

To conclude we consider d-graceful α-labelings for ladders. We recall that the *ladder graph* of order $2k$, denoted by L_{2k}, can be seen as the cartesian product of the path $P_2 = (0 \sim 1)$ of length 1 by the path $P_k = (0 \sim 1 \sim \ldots \sim k-1)$ of length $k-1$. The ladder graph L_{2k} is clearly bipartite with bipartite sets A_0, A_1 where A_h is the set of pairs $(i, j) \in \{0, 1\} \times \{0, 1, \ldots, k-1\}$ such that $i + j$ has the same parity as h.

Theorem 4.5 *The ladder graph L_{2k} has a 2-graceful α-labeling if and only if k is even.*

Proof. Since L_{2k} has $e = 3k - 2$ edges, 2 divides e if and only if k is even. Suppose now k to be even. One can check that a 2-graceful α-labeling of L_{2k} is given by

$$f(i,j) = \begin{cases} j \text{ if } (i,j) \in A_0 \\ 3k - 2j - 1 \text{ if } (i,j) \in A_1, \ j = 0, \ldots, \frac{k}{2} - 1 \\ 3k - 2j - 2 \text{ if } (i,j) \in A_1, \ j = \frac{k}{2}, \ldots, k - 1. \end{cases}$$

$\square$

We observe that the 2-graceful α-labeling constructed in the previous theorem is very similar to the α-labeling of L_{2k} proposed by the author in [12].

Figure 6 shows the 2-graceful α-labeling of L_{16} provided by Theorem 4.5.
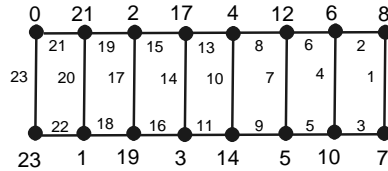

Figure 6: L_{16}

As an immediate consequence of Theorem 2.6 we have

Proposition 4.6 *Let k be an even integer. There exists a cyclic L_{2k}-decomposition of $K_{\frac{3k}{2} \times 4n}$ for any integer $n \geq 1$.*

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